# A Density Independent formulation of Smoothed Particle Hydrodynamics

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ELSI

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# About Me


- Takayuki Saitoh (斎藤貴之)
- 1977/02/18
- Born in Asahikawa
- Career
  - 2004/3 PhD Hokkaido Univ.
  - 2004/4-2011/7 Researcher-JSPJ fellow-Research specialist, NAOJ
  - 2011/8-2012/1 Researcher, Titech
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- Research interests
  - Galaxy formation
  - High performance computing, parallel computing, algorithms

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# A DENSITY-INDEPENDENT FORMULATION OF SMOOTHED PARTICLE HYDRODYNAMICS

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## ABSTRACT

The standard formulation of the smoothed particle hydrodynamics (SPH) assumes that the local density distribution is differentiable. This assumption is used to derive the spatial derivatives of other quantities. However, this assumption breaks down at the contact discontinuity. At the contact discontinuity, the density of the low-density side is overestimated while that of the high-density side is underestimated. As a result, the pressure of the low-density (high-density) side is overestimated (underestimated). Thus, unphysical repulsive force appears at the contact discontinuity, resulting in the effective surface tension. This tension suppresses fluid instabilities. In this paper, we present a new formulation of SPH, which does not require the differentiability of density. Instead of the mass density, we adopt the internal energy density (pressure) and its arbitrary function, which are smoothed quantities at the contact discontinuity, as the volume element used for the kernel integration. We call this new formulation density-independent SPH (DISPH). It handles the contact discontinuity without numerical problems. The results of standard tests such as the shock tube, Kelvin–Helmholtz and Rayleigh–Taylor instabilities, point-like explosion, and blob tests are all very favorable to DISPH. We conclude that DISPH solved most of the known difficulties of the standard SPH, without introducing additional numerical diffusion or breaking the exact force symmetry or energy conservation. Our new SPH includes the formulation proposed by Ritchie & Thomas as a special case. Our formulation can be extended to handle a non-ideal gas easily.

*Key words:* galaxies: evolution – galaxies: ISM – methods: numerical

# Astrophysical Gas

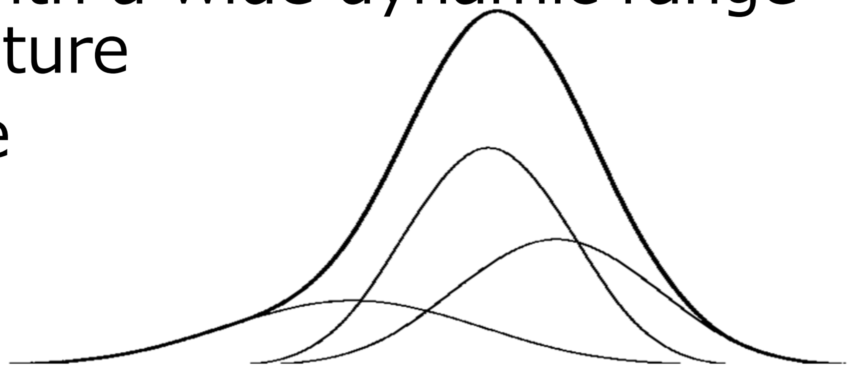
- Very high density contrast
  - $10^{-30}$ g/cc ← cosmic mean
  - $10^{-22}$ g/cc ← star forming gas
  - 150 g/cc ← Sun's core
- Self-gravity
- Compressibility
  - Shock is ubiquitous
- High Reynolds number
  - turbulence
- Radiation/Magnetic field

# Governing equations

- Continuity equation $$\frac{d\rho}{dt} = -\rho\nabla\cdot\boldsymbol{v}$$
- Momentum equation $$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{\nabla P}{\rho}$$
- Energy equation $$\frac{du}{dt} = -\frac{P}{\rho}\nabla\cdot\boldsymbol{v}$$
- Equation of state $$P = (\gamma - 1)\rho u$$

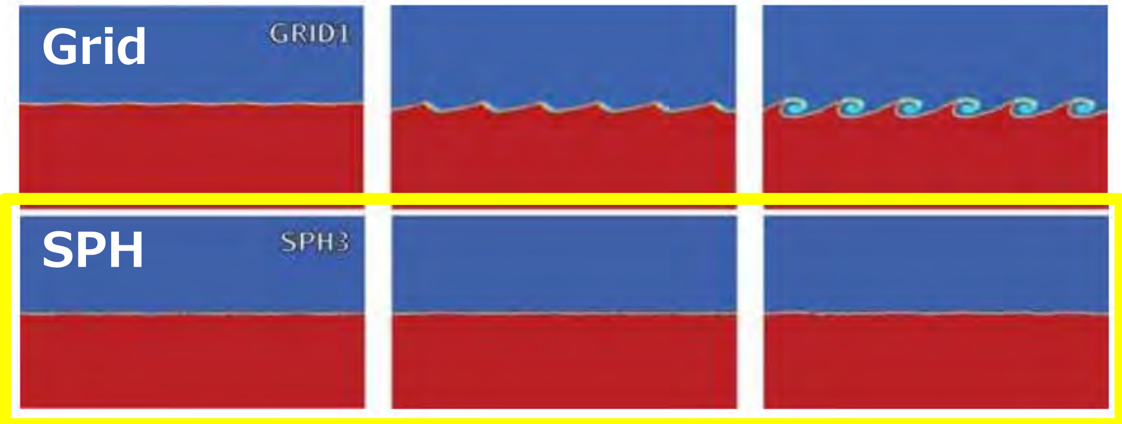
# Smoothed Particle Hydrodynamics

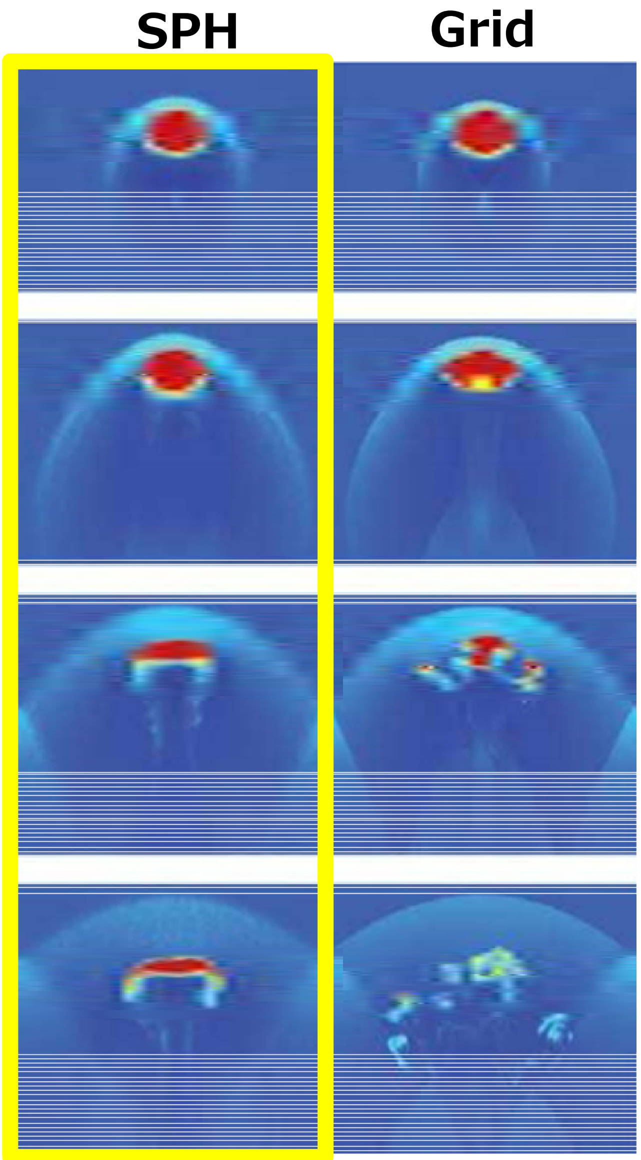
- Lagrangian scheme of fluid dynamics developed by Lucy (1977) and Gingold & Monaghan (1977)
  - Solve evolution of fluid elements
  - Fluid quantities are evaluated via the convolution of particles
- Advantages
  - Galilean invariance
  - Suitable for simulations with a wide dynamic range because of Lagrangian nature
  - High density regions have high resolution



# Problem in SPH

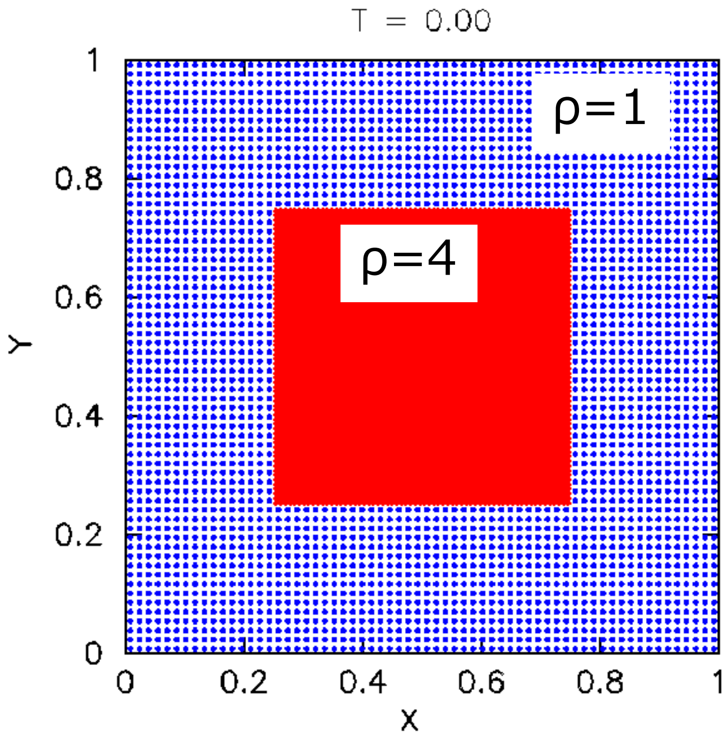
- SPH cannot deal with *contact discontinuities*, resulting in suppression of fluid instabilities (Agertz+2007)
  - The reason is that the standard formulation of SPH uses differentiability of density

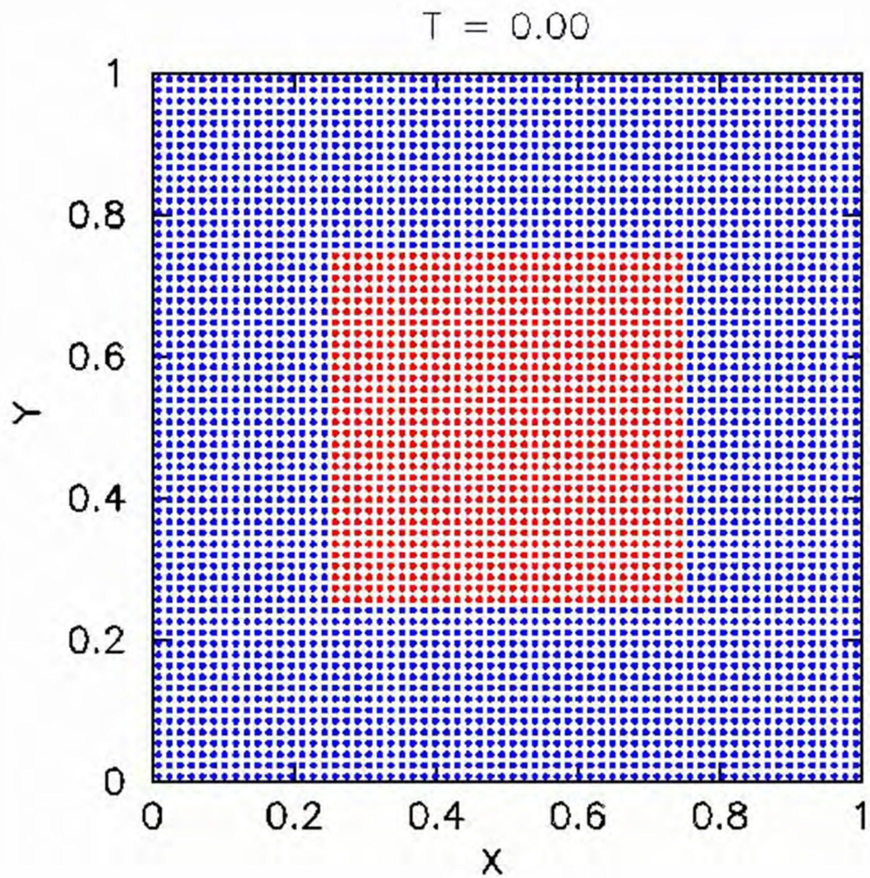


# Hydrostatic test

- Initially pressure equilibrium fluid
- By definition, the fluid should be static



# Derivation of SPH (1)

- Physical quantity, *f*, at *x* is

$$f(\boldsymbol{r}) = \int f(\boldsymbol{r}')\delta(|\boldsymbol{r} - \boldsymbol{r}'|)d\boldsymbol{r}'.$$

- Applying the kernel approximation, this equation becomes

$$\langle f \rangle(\boldsymbol{r}) = \int f(\boldsymbol{r}')W(|\boldsymbol{r} - \boldsymbol{r}'|, h)dx',$$

  where *W* is the compact support function
- Spatial derivation is

$$\langle \nabla f \rangle(\boldsymbol{r}) = \int \nabla f(\boldsymbol{r}')W(|\boldsymbol{r} - \boldsymbol{r}'|, h)d\boldsymbol{r}'.$$

$$\Rightarrow \quad \langle \nabla f \rangle(\boldsymbol{r}) = \int f(\boldsymbol{r}')\nabla W(|\boldsymbol{r} - \boldsymbol{r}'|, h)d\boldsymbol{r}'.$$

# Derivation of SPH (2)

- Discretize the kernel approximated equations using the volume element **dr' = m/ρ**
  ➔
$$\langle f\rangle(\boldsymbol{r}_i) \simeq \sum_j m_j \frac{f_j}{\rho_j} W(r_{ij}, h),$$

- Substituting $f = \rho$, we have

$$\boxed{\rho_i \simeq \sum_j m_j W(r_{ij}, h),}$$

<u>In standard SPH, every quantities evaluate using this $\rho$</u>

# Derivation of SPH (3)

- Euler eq. $$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{\nabla P}{\rho},$$

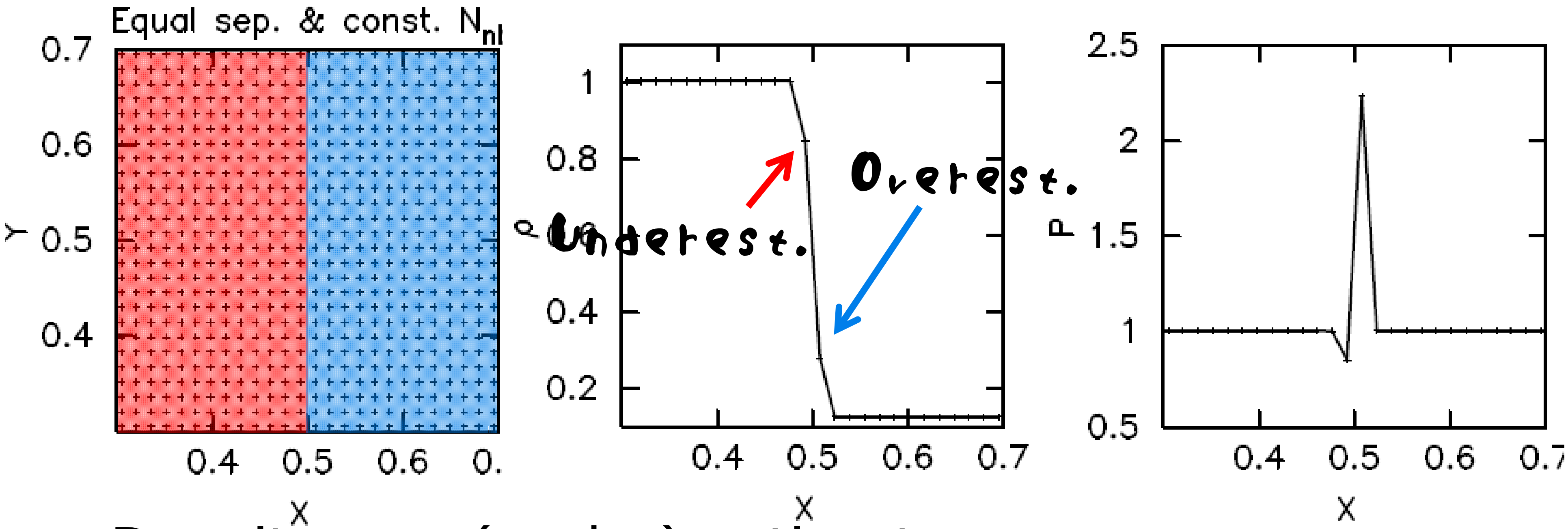
$$\boxed{\frac{d^2\boldsymbol{r}_i}{dt^2} \simeq -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right) \nabla W(r_{ij}, h).}$$ $$\left(\frac{\nabla P}{\rho} = \nabla\left(\frac{P}{\rho}\right) - \frac{P}{\rho^2}\nabla\rho.\right)$$

- Energy eq. $$\frac{du}{dt} = -\frac{P}{\rho}\nabla \cdot \boldsymbol{v},$$

$$\boxed{\frac{du_i}{dt} \simeq \sum m_j \frac{P_i}{\rho_i^2} \boldsymbol{v}_{ij} \cdot \nabla W(r_{ij}, h).}$$ $$\left(\nabla(\rho\boldsymbol{v}) = \nabla\rho\boldsymbol{v} + \rho\nabla\cdot\boldsymbol{v}.\right)$$

- Equation of State: $P = (\gamma-1)\rho u$
- **Differentiability of density is used.**

# Pressure at Contact Discon.




- Density over(under) estimate
  ➔ Error in pressure (=repulsive force)
  ➔ Suppression of mixing
- **We should reconstruct SPH with different way in order to avoid differentiability of density**

# Density Independent SPH (1)

- We use new volume element for discretization:

$$d\boldsymbol{r} = \frac{(\gamma - 1)mu}{P}.$$

  - Physical quantity, f, is

$$\langle f\rangle(\boldsymbol{r}) \simeq \sum_j (\gamma - 1)\frac{U_j f_j}{P_j} W(r_{ij}, h_i), \quad U_j = m_j u_j$$

  - Corresponding fundamental equation is

$$\boxed{q_i \simeq \sum_j U_j W(r_{ij}, h_i).}$$

The value *q* is proportional to pressure in an ideal-gas

# Density Independent SPH (2)

- Euler eq. $$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{\nabla P}{\rho},$$

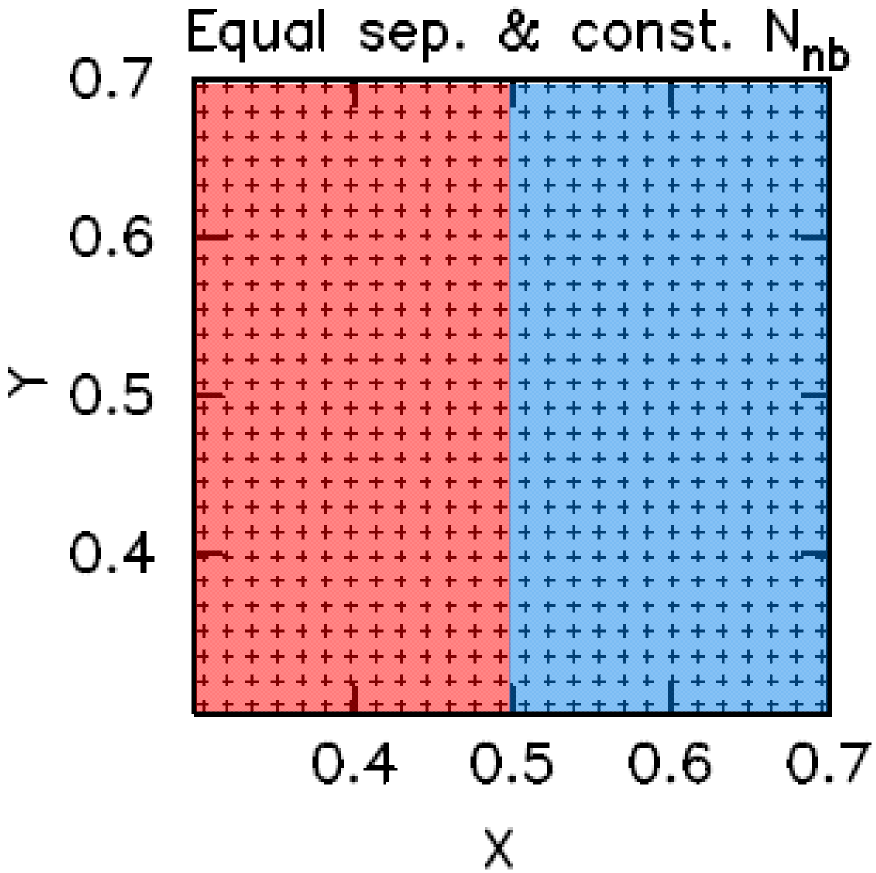
$$\boxed{m_i \frac{d\boldsymbol{v}_i}{dt} \simeq -(\gamma - 1)\sum_j U_i U_j \left(\frac{1}{q_i} + \frac{1}{q_j}\right) \nabla \tilde{W}_{ij}}$$

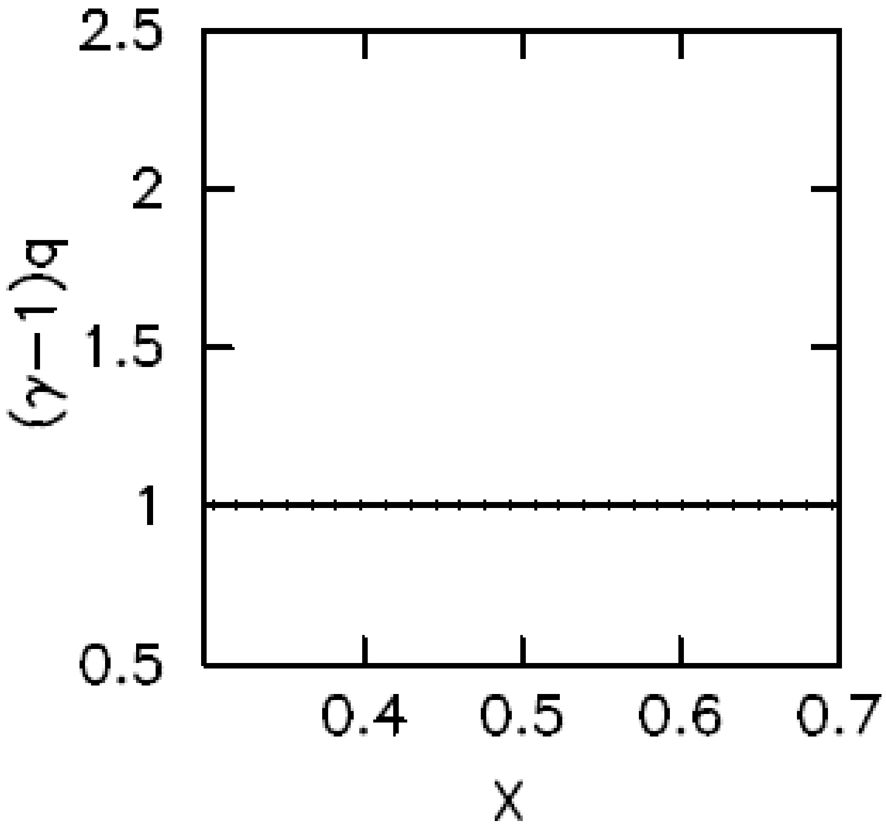
- Energy eq. $$\frac{du}{dt} = -\frac{P}{\rho}\nabla \cdot \boldsymbol{v},$$

$$\boxed{\frac{dU_i}{dt} \simeq (\gamma - 1)\sum_j \frac{U_i U_j}{q_i} \boldsymbol{v}_{ij} \cdot \nabla \tilde{W}(r_{ij}, h_{ij}),}$$

- We have fluid equations without density in their right-hand-sides.

# Pressure at Contact Discon.



# Summary of Equations

- Standard SPH

$$\rho_i \simeq \sum_j m_j W(r_{ij}, h),$$

$$\frac{d^2 \boldsymbol{r}_i}{dt^2} \simeq -\sum_j m_j \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla W(r_{ij}, h).$$

$$\frac{du_i}{dt} \simeq \sum m_j \frac{P_i}{\rho_i^2} \boldsymbol{v}_{ij} \cdot \nabla W(r_{ij}, h).$$

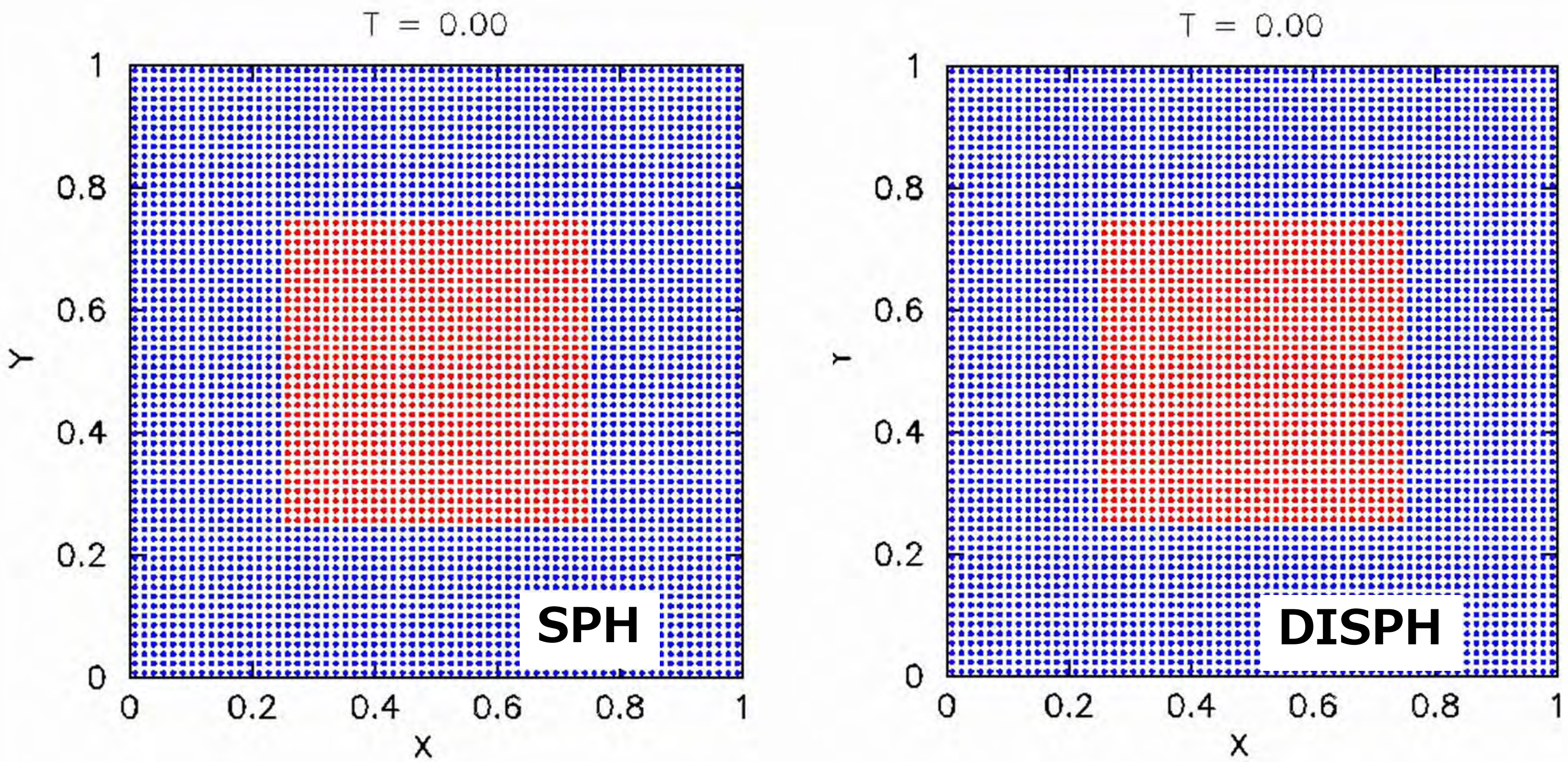
- DISPH

$$q_i \simeq \sum_j U_j W(r_{ij}, h_i).$$

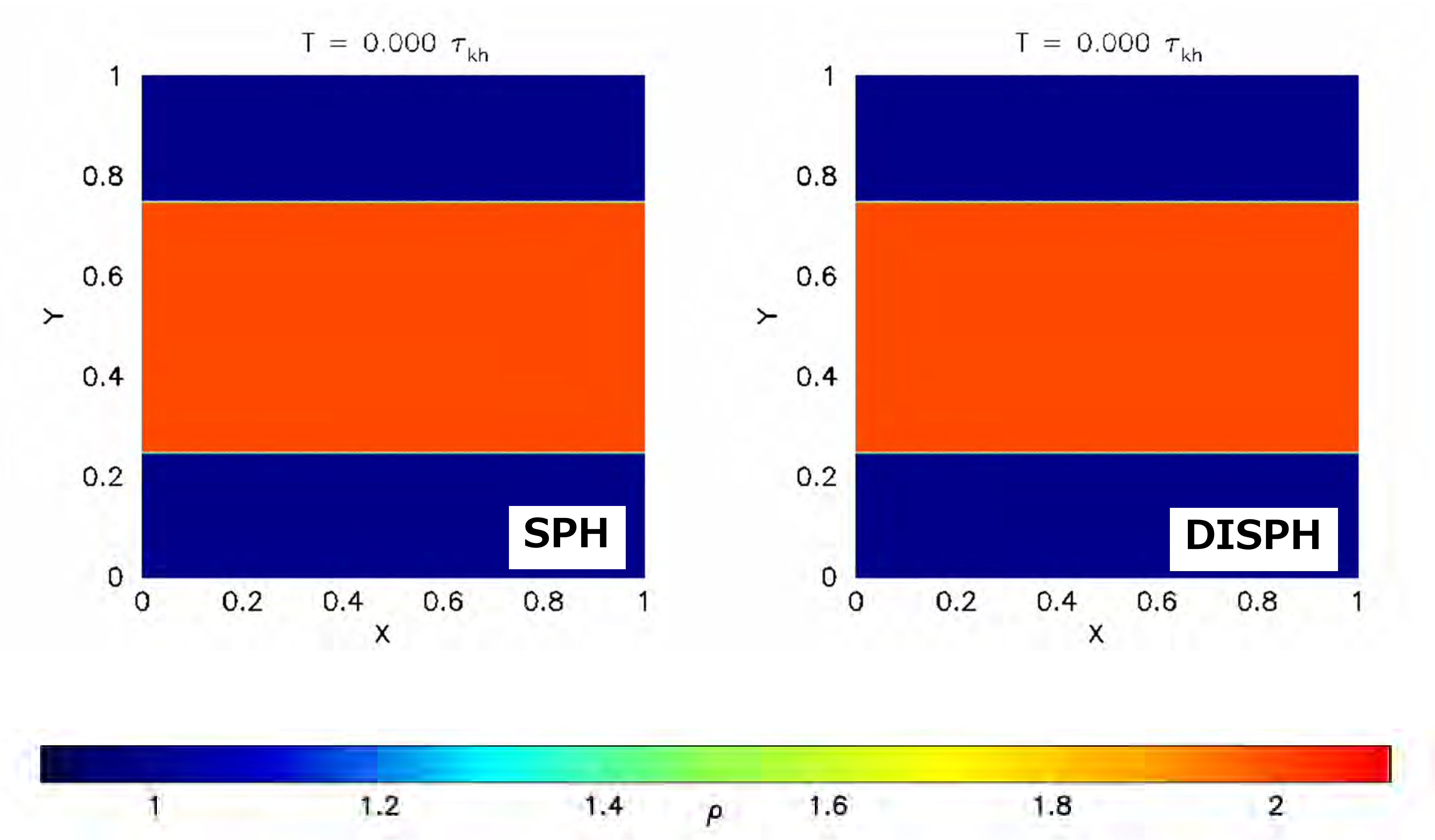
$$m_i \frac{d\boldsymbol{v}_i}{dt} \simeq -(\gamma - 1) \sum_j U_i U_j \left( \frac{1}{q_i} + \frac{1}{q_j} \right) \nabla \tilde{W}_{ij}$$

$$\frac{dU_i}{dt} \simeq (\gamma - 1) \sum_j \frac{U_i U_j}{q_i} \boldsymbol{v}_{ij} \cdot \nabla \tilde{W}(r_{ij}, h_{ij}),$$

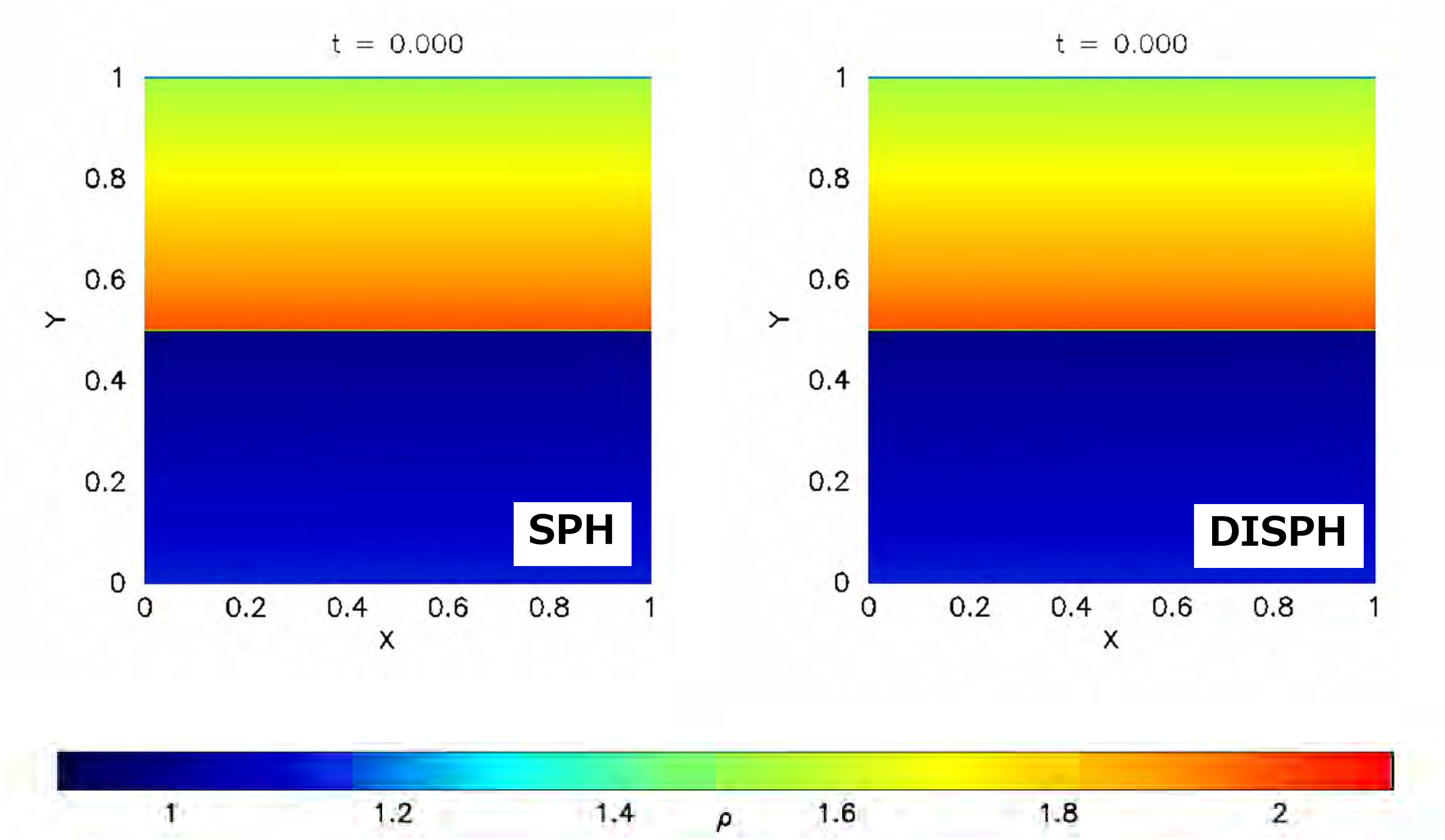
# Hydrostatic tests



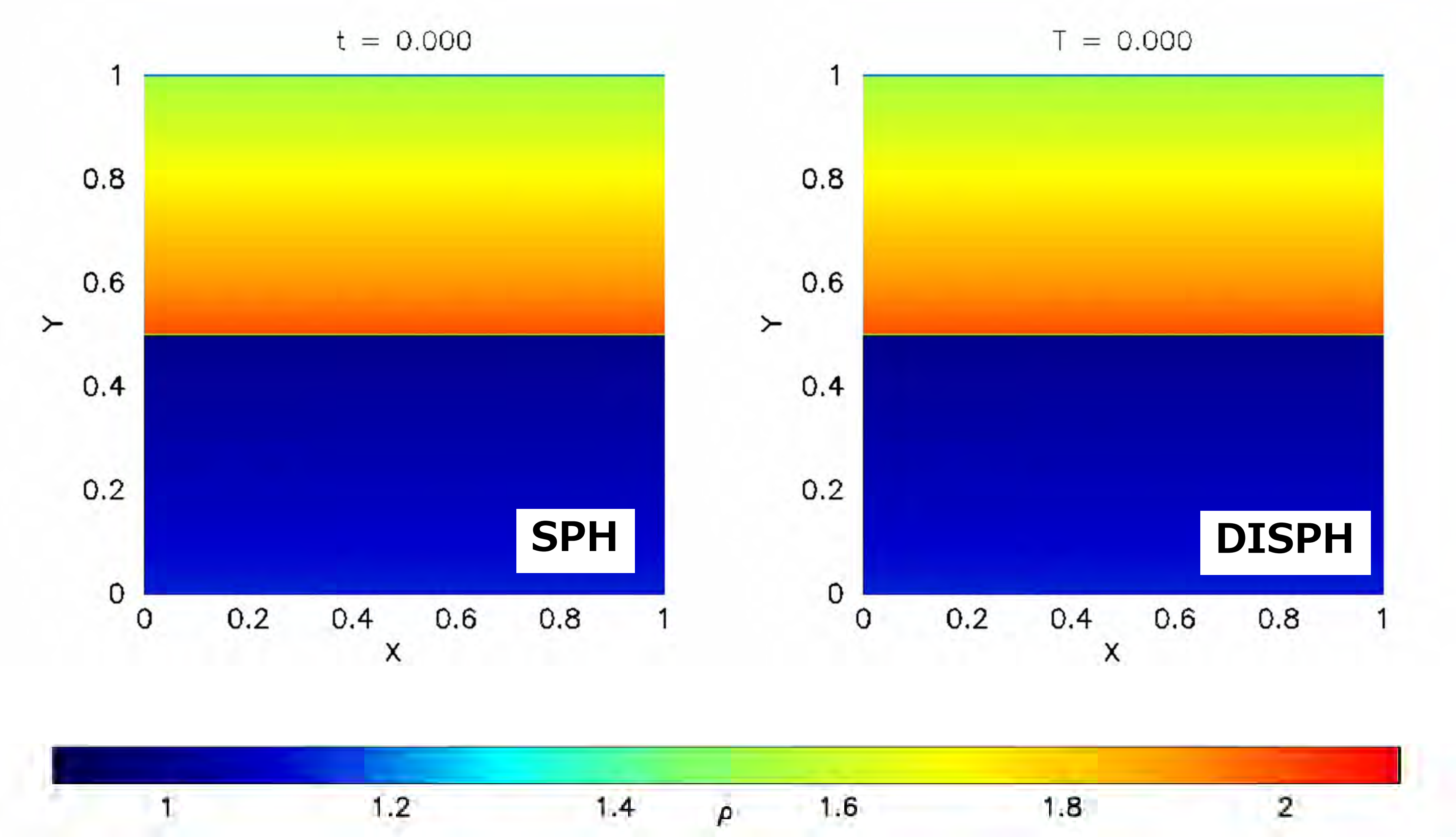
# Kelvin-Helmholtz instability tests

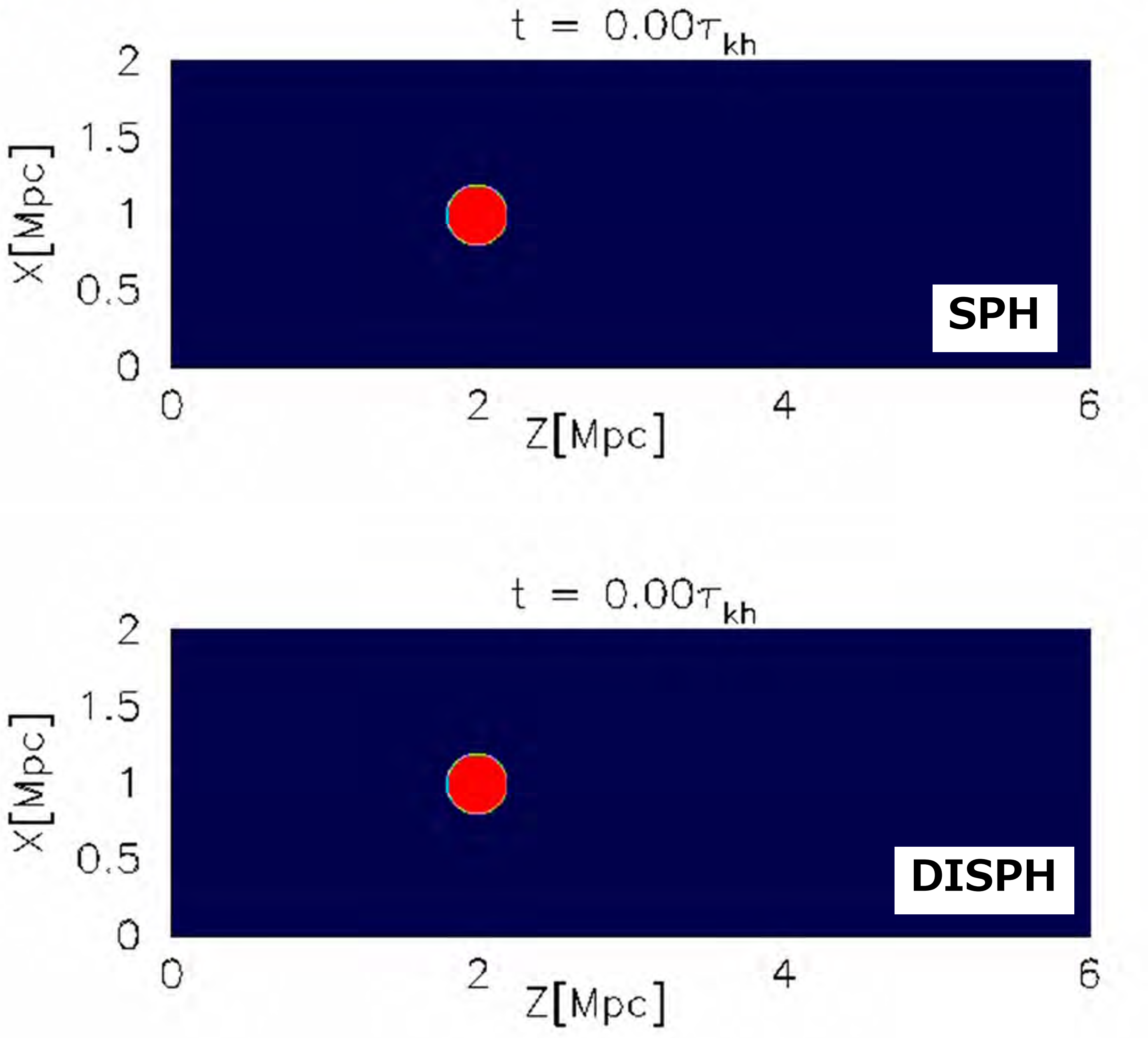


# Rayleigh-Taylor instability tests

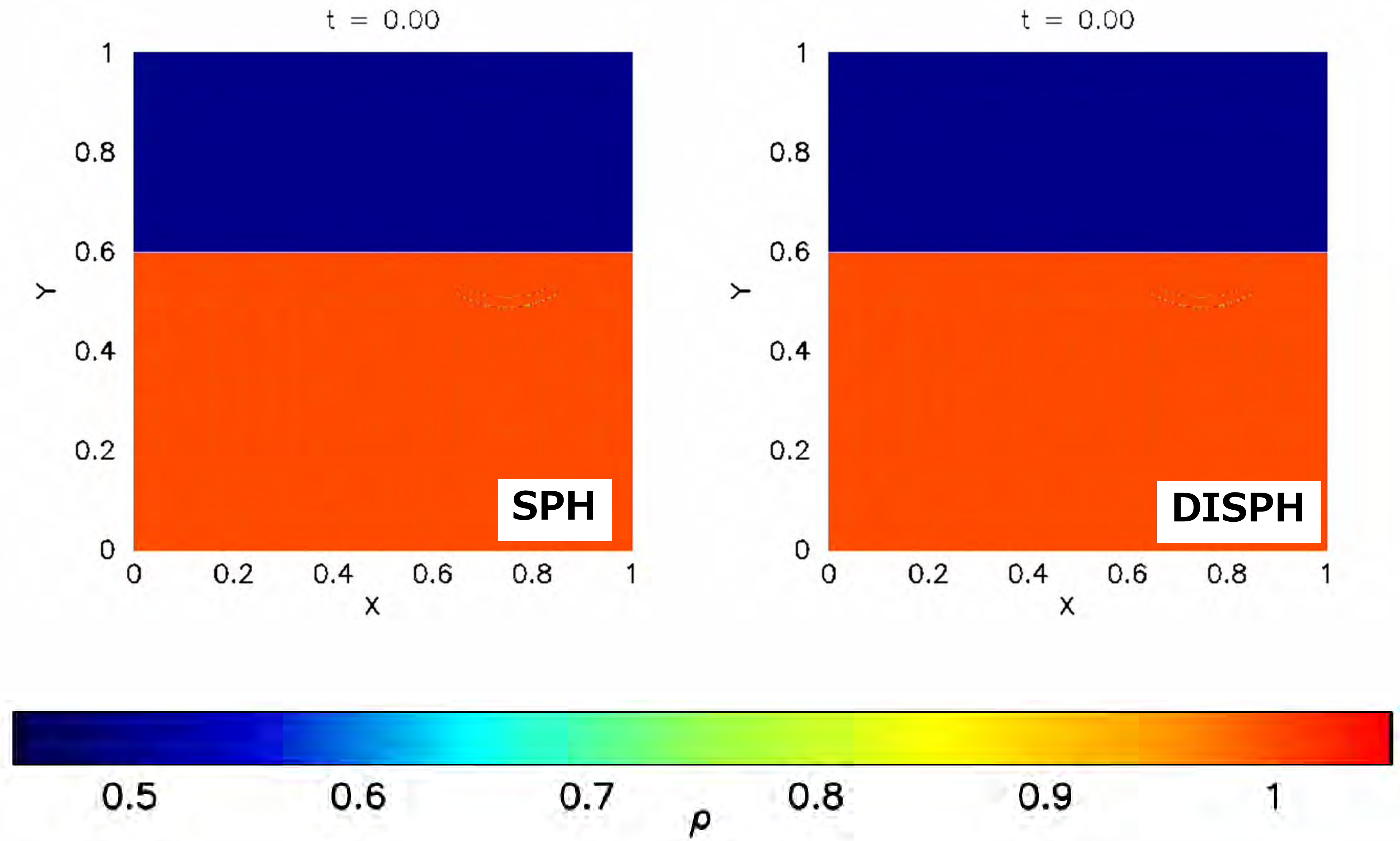


# Rayleigh-Taylor instability tests




t = 0.00τ_kh
2
1.5
1
0.5
0
X[Mpc]
0
2
4
6
Z[Mpc]
SPH
t = 0.00τ_kh
2
1.5
1
0.5
0
X[Mpc]
0
2
4
6
Z[Mpc]
DISPH

# Two phase fluid mixing



# Lagrangian Formulation (1)

- Lagrangian eq.

$$L(\dot{\boldsymbol{Q}}, \boldsymbol{Q}) = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{Q}}_i^2 - \sum_i U_i(\boldsymbol{Q})$$

$$\boldsymbol{Q} \equiv (\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, h_1, h_2, \ldots, h_N)$$

- A constraint eq.

$$\phi_i = \frac{4\pi}{3}(2h_i)^3 \frac{q_i}{U_i} - N_{\rm ngb} = 0.$$

- E-L eq. with a constraint eq.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{Q}}_i} - \frac{\partial L}{\partial \boldsymbol{Q}_i} = \sum_j \lambda_j \frac{\partial \phi_j}{\partial \boldsymbol{Q}_i}$$

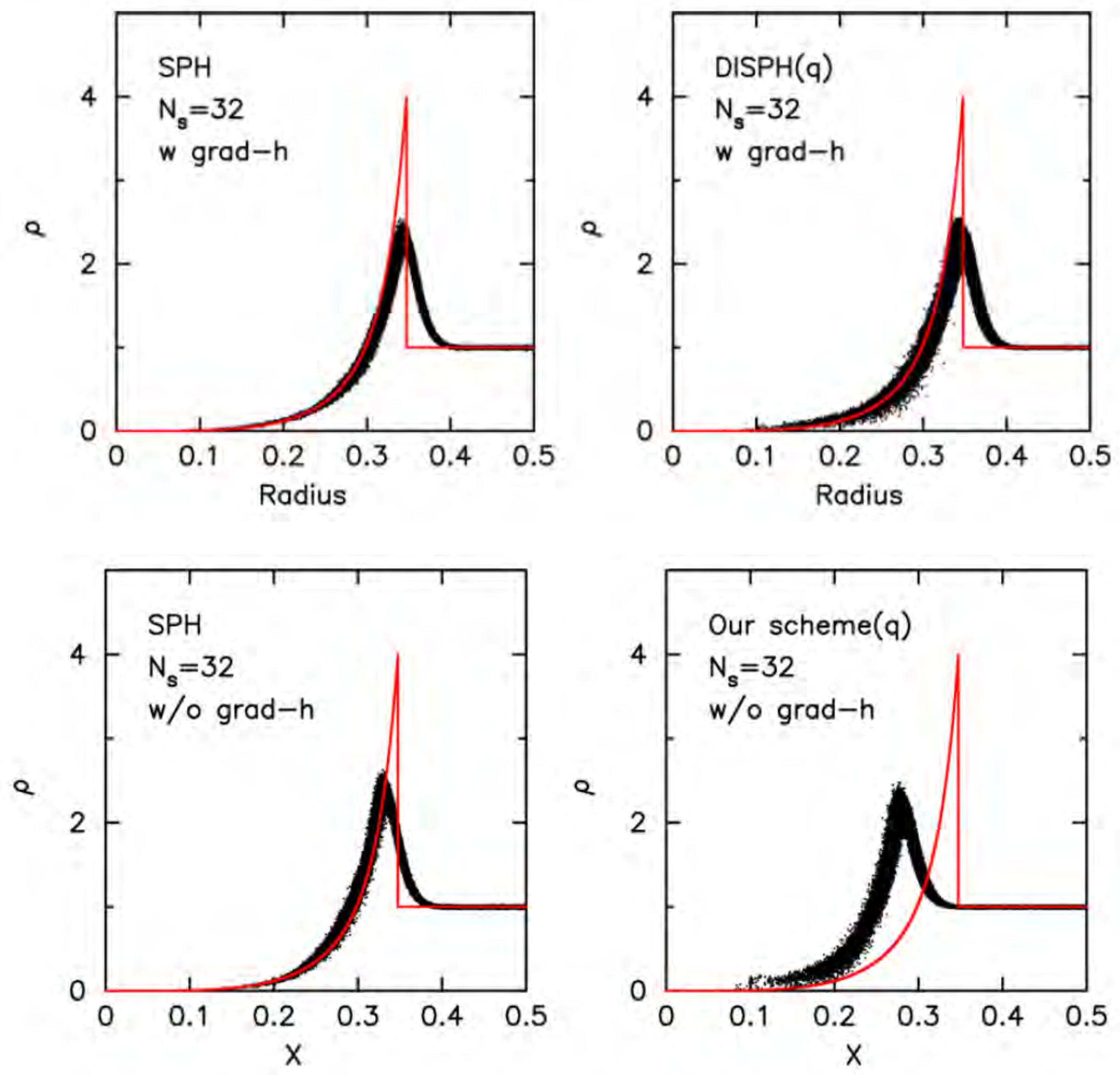
# Lagrangian Formulation (2)

$$m_i \frac{dv_i}{dt} = -(\gamma - 1) \sum_j U_i U_j \left( \frac{1}{q_i} f_i^{\mathrm{grad}} \nabla_i W_{ij}(h_i) + \frac{1}{q_j} f_j^{\mathrm{grad}} \nabla_i W_{ij}(h_j) \right)$$

$$\frac{dU_i}{dt} = (\gamma - 1) \sum_j \frac{U_i U_j}{q_i} f_i^{\mathrm{grad}} \boldsymbol{v}_{ij} \cdot \nabla_i W_{ij}(h_i)$$

$$f_j^{\mathrm{grad}} = \left( 1 + \frac{h_j}{3q_j} \frac{\partial q_j}{\partial h_j} \right)^{-1}$$

# Point like explosion



# Generalized DISPH (1)

- Unlike q ($\propto$P), we use $y=P^{\zeta}$ as a fundamental quantity and use $\zeta < 1$

$$\langle y \rangle(\boldsymbol{r}_i) = \sum_j Z_j W(r_{ij}, h)$$

$$\frac{dU_i}{dt} = \frac{P_i}{y_i^2} \sum_j Z_i Z_j \boldsymbol{v}_{ij} \cdot \nabla \tilde{W}_{ij}$$

$$m_i \frac{d\boldsymbol{v}_i}{dt} = -\sum_j Z_i Z_j \left( \frac{P_i}{y_i^2} + \frac{P_j}{y_j^2} \right) \nabla \tilde{W}_{ij}$$

- This generalized form is adequate for the strong shocks, like SN explosion

# Generalized DISPH (2)

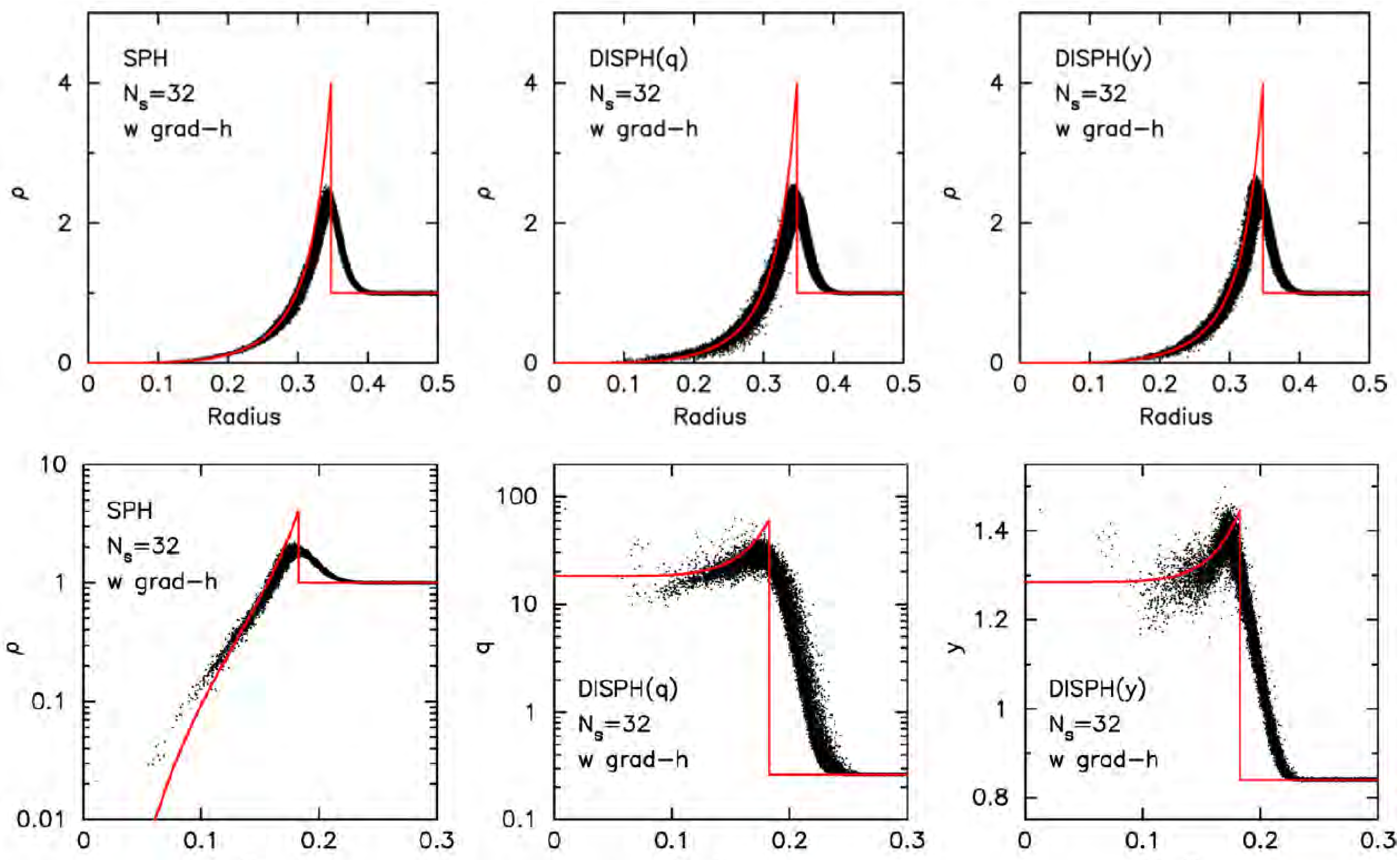
$$m_i \frac{d\boldsymbol{v}_i}{dt} = -\sum_j Z_i Z_j \times \left( \frac{P_i}{y_i^2} f_i^{\mathrm{grad}} \nabla_i W_{ij}(h_i) + \frac{P_j}{y_j^2} f_j^{\mathrm{grad}} \nabla_i W_{ij}(h_j) \right)$$

$$\frac{dU_i}{dt} = \frac{P_i Z_i}{y_i^2} f_i^{\mathrm{grad}} \sum_j Z_j \boldsymbol{v}_{ij} \cdot \nabla_i W_{ij}(h_i)$$

$$\frac{dZ_i}{dt} = (\zeta(P_i)\gamma - 1) f_i^{\mathrm{grad}} \sum \frac{Z_i Z_j}{y_i} \boldsymbol{v}_{ij} \cdot \nabla W_{ij}(h_i)$$

$$f_i^{\mathrm{grad}} = \left( 1 + \frac{h_i}{3y_i} \frac{\partial y_i}{\partial h_i} \right)^{-1}$$

# Generalized DISPH (3)



# Extension for Non-ideal Gas

- For geoscience, non-ideal EOSs are essential to express mantle, rock, iron cores...
- Recently, Hosono, Saitoh, & Makino have developed an extension of DISPH for non-ideal gas
  - P=P(ρ,U), where P consists of complex functions/tables
  - Giant impact simulations
  - PASJ Accepted! astro-ph:1307.0916

16th, Aug, 2013

# Summary

- We have developed a new SPH which uses energy density ($\propto$ pressure) as a basis of the formulation
- Advantages
  - Holds all advantages equipped with the standard SPH
  - Can deal with contact discontinuities, resulting in the growth of the fluid instabilities
  - Equations are quite simple